MATH 118: Final

Name: Key

Directions:

* Show your thought process (commonly said as ”show your work”) when solving each problem for full credit.
* If you do not know how to solve a problem, try your best and/or explain in English what you would do.
* Good luck!

Problem	Score	Points
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
		120

1. Short answer questions:

(a) Suppose you try distributing

$$(x+y)^2 z^2 = (xz^2 + yz^2)^2$$

Why is this incorrect?

Because $(x+y)^2 z^2 = (x+y)(x+y)z^2$; you distributed the z^2 into both factors of $(x+y)$. The distributive law says you can only distribute into one factor of $(x+y)$.

(b) Suppose you cancel out the x's to simplify

$$\frac{3+x}{x} = \frac{3+1}{1} = 4$$

Why is this incorrect?

Because x is a term in the context of the numerator. Fraction law #5 says $\frac{a \cdot \cancel{c}}{b \cdot \cancel{c}} = \frac{a}{b}$; you can only cancel if c is a factor in the context of the entire numerator and denominator.

(c) You try simplifying by distributing:

$$[(x-1)^2 + (x+1)]^3 = (x-1)^5 + (x+1)^3 = x^5 - 1^5 + x^3 + 1^3 = x^5 + x^3$$

Circle the two types of mistakes you made and explain why they are mistakes.

In purple:

① Distributed the power of 3 to the term $(x-1)^2$

② If you had $((x-1)^2)^3$, you need to multiply 2 and 3 not add them.

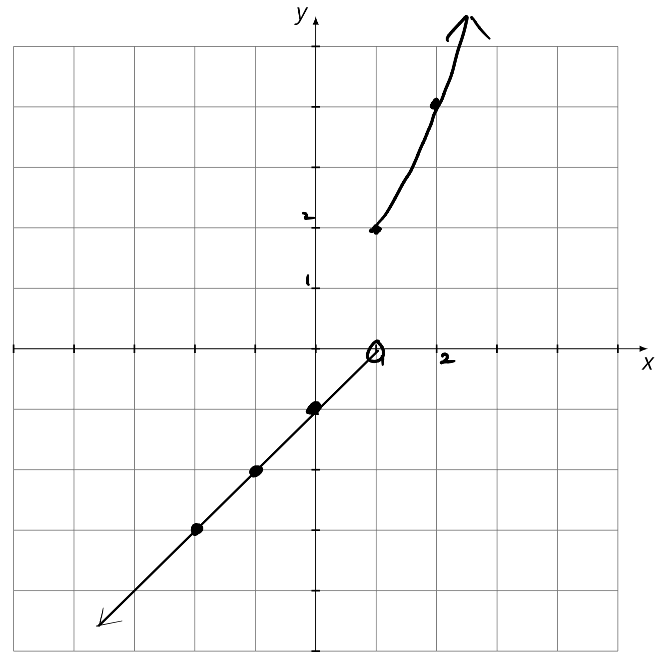
In red: Distributed exponent to terms. Can only distribute to factors!

2. Suppose

$$f(x) = \begin{cases} x - 1 & x < 1 \\ 2^x & x \geq 1 \end{cases}$$

x	$f(x)$
-2	-3
-1	-2
0	-1
1	2
2	4

(a) Sketch a graph of $f(x)$.



(b) What is $f(1)$?

$f(1) = 2^1 = 2$

3. Isolate the variable in the following equations:

(a) $4x + 2 = 6x - w$, for x

$$-4x + w \qquad -4x + w$$

$$\frac{1}{2}(2 + w) = \cancel{2}x \cdot \frac{1}{\cancel{2}}$$

$$\boxed{x = \frac{1}{2}(2 + w)}$$

get x out of denominator

(b) $\dfrac{2x+1}{x-2} = 1$, for x.

$$\cancel{(x-2)} \cdot \frac{2x+1}{\cancel{x-2}} = 1 \cdot (x-2)$$

$$2x + 1 = x - 2$$

$$-x - 1 \qquad -x - 1$$

$$\boxed{x = -3}$$

(c) $4x(y-1) - 6x(z+2) - w(4xz - y + 1) = x$, for x

Make terms with x then gather on one side.
Then make x a factor by using GCF.

$$4xy - 4x - 6xz - 12x - 4xwz + wy - w = x$$

$$-x \quad -wy + w \qquad -x - wy + w$$

$$4xy - 4x - 6xz - 12x - 4xwz - x = w - wy$$

$$x(4y - 4 - 6z - 12 - 4wz - 1) = w - wy$$

GCF

$$x \cdot (4y - 6z - 4wz - 17) = w - wy$$

combine like terms

x is a factor.
get rid of other factor by dividing.

$$\boxed{x = \frac{w - wy}{4y - 6z - 4wz - 17}}$$

4. Suppose $f(x) = x - x^2$

(a) A person tries to find $f(x+h)$ by writing

$$f(x+h) = x - x^2 + h$$

This is wrong. What expression (involving $f(x)$) did the person actually write down?

$$\underbrace{x - x^2}_{f(x)} \underbrace{+ h}_{+h} \qquad \text{so } \boxed{f(x) + h}$$

(b) The person then tries again:

$$f(x+h) = x + h - x + h^2$$

Explain the reason why this is also incorrect.

$x+h$ is two terms. In $f(x) = x - x^2$, replacing the x in $-x^2$ with $x+h$ requires parenthesis since two terms are being subtracted and taken to a power.

(c) Your turn: Evaluate $f(x+h)$ and fully simplify.

$$f(x+h) = (x+h) - (x+h)^2$$

this is often forgotten as well. You are subtracting the entirety of three terms.

$$\overset{\text{special product}}{=} x + h - (x^2 + 2xh + h^2)$$

$$= \boxed{x + h - x^2 - 2xh - h^2}$$

(d) In general, when you are substituting two or more terms into **(a)** a variable with a power or **(b)** that variable being subtracted, what do you need to not forget?

Do not forget to put parenthesis around the entire expression you are subtracting or taking the power of.

5. Solve the equation for x. Check your work if necessary.

(a) $x^2 = 3(x-1)$ Quadratic. Put into $ax^2+bx+c=0$.

$$x^2 = 3x - 3 \quad \text{dist law}$$
$$\underline{-3x+3} \qquad \underline{-3x+3}$$

$$x^2 - 3x + 3 = 0$$

$a=1,\ b=-3,\ c=3$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$= \frac{-(-3) \pm \sqrt{(-3)^2 - 4\cdot 1\cdot 3}}{2\cdot 1}$$
$$= \frac{3 \pm \sqrt{9-12}}{2}$$
$$= \frac{3 \pm \sqrt{-3}}{2}$$
$$= \frac{3 \pm i\sqrt{3}}{2}$$

$$\boxed{x = \frac{3 \pm i\sqrt{3}}{2}}$$

(b) $\dfrac{1}{x-1} - \dfrac{2}{x^2} = 0$

Get x out of the denominator. Remember: goal is $x = \cdots$ not in denominator.

$$(x-1)\cdot x^2 \cdot \underbrace{\left(\frac{1}{x-1} - \frac{2}{x^2}\right)}_{\text{this is two terms.}} = 0 \cdot x^2 \cdot (x-1)$$

do not forget parentheses.

$$\cancel{(x-1)}x^2 \frac{1}{\cancel{x-1}} - (x-1)\cancel{x^2}\frac{2}{\cancel{x^2}} = 0 \quad \text{Distributive law}$$

$$x^2 - 2(x-1) = 0 \quad \text{Fraction law \#1 then \#5}$$

$$x^2 - 2x + 2 = 0 \quad \text{Distributive law}$$

$$\boxed{\text{answer: } x = 1 \pm i}$$

$a=1,\ b=-2,\ c=2$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4\cdot 1\cdot 2}}{2} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm i\sqrt{4}}{2}$$

terms

$$= \frac{2 \pm 2i}{2} \underset{\text{GCF}}{=} \frac{\cancel{2}\cdot(1 \pm i)}{\cancel{2}} \underset{\text{frac law \#5}}{=} 1 \pm i$$

6. Solve the equation for x. Check your work if necessary.

(a) $x^2e^x + xe^x - e^x = 0$

$e^x(x^2 + x - 1) = 0$ GCF

$e^x = 0$, $x^2 + x - 1 = 0$ Zero Product Property

No solution since range of e^x is $(0, \infty)$

$a = 1,\ b = 1,\ c = -1$

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1}$$

$$= \frac{-1 \pm \sqrt{1 + 4}}{2}$$

$$= \frac{-1 \pm \sqrt{5}}{2}$$

answer: $\boxed{x = \frac{-1 \pm \sqrt{5}}{2}}$

(b) $2 \log x = \log 2 + \log(3x - 4)$

We use the $\log_a x = \log_a y \implies x = y$ technique.

$\log x^2 = \log(2(3x - 4))$ Laws of logs #1 and #3

quadratic. put into $ax^2 + bx + c = 0$

$x^2 = 2(3x - 4)$ Because log is 1-1

$x^2 - 2(3x - 4) = 0$

$x^2 - 6x + 8 = 0$ Dist law

$-2 - 4 = -6$ ✓

$(x - 4)(x - 2) = 0$

$\boxed{x = 4,\ x = 2}$

7. Fully simplify each expression; **your answer should be a number**. No fractional expressions or negative exponents.

   (a) $\log_2 2 = 1$

   (b) $e^0 = 1$

   (c) $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = \sqrt[3]{4^3} = 4^{\frac{3}{3}} = 4$

   (d) $\log_{10} 2 + \log_{10} 5 = \log_{10}(2 \cdot 5) = \log_{10} 10 = 1$

   (e) $\frac{5}{12} + \frac{13}{16} = \frac{4}{4} \cdot \frac{5}{12} + \frac{13}{16} \cdot \frac{3}{3} = \frac{20}{48} + \frac{26}{48} = \frac{46}{48} = \frac{\cancel{2} \cdot 23}{\cancel{2} \cdot 24}$

   $= \frac{23}{24}$

   $12 = 2 \cdot 2 \cdot \boxed{3}$

   $16 = 2 \cdot 2 \cdot \boxed{2 \cdot 2}$

   (f) $(1+i)(1-i) \overset{\text{dist}}{\underset{\text{law}}{=}} (1+i) \cdot 1 - (1+i) \cdot i$

   $= 1 + i - i - i^2$

   $= 1 - (-1) = 1 + 1 = 2$

   (g) i^4

   $= (i^2)^2 = (-1)^2 = 1$

   (h) $-2^4 = -16$

   (i) $2^{-5} \cdot 2^4 \cdot \frac{1}{2^3} \overset{\text{L.E}}{=} 2^{-5+4-3} = 2^{-4} = \frac{1}{2^4} = \frac{1}{16}$

8. Completely factor each expression. Try to obtain a complete factorization.

(a) $16a^2 - 24a + 9$ quadratic

$$\begin{array}{cc} 4 & -3 \\ 4 & -3 \end{array}$$

$4(-3) + 4(-3) = -24$ ✓

$$\boxed{(4a-3)^2}$$

(b) $(x^3 + 4x^2) + (x + 4)$ grouping $= x^2(x+4) + (x+4)$

$= (x+4)(x^2+1)$

solve $x^2+1=0$

$x^2=-1$

$x = \pm\sqrt{-1} = \pm i$

By factor theorem

$x^2+1 = (x-i)(x+i)$

$= \boxed{(x+4)(x+i)(x-i)}$

(c) $(x + y)^2 - 2(x + y) + 1$

Let $z = (x+y)$.

quadratic $\rightarrow$ $z^2 - 2z + 1 = (z-1)^2 = \boxed{(x+y-1)^2}$

$$\begin{array}{cc} 1 & -1 \\ 1 & -1 \end{array}$$

(d) $(x - 1)(x + 2)^2 - (x - 1)^2(x + 2)$

$\overset{GCF}{=} (x-1)(x+2)(x+2-(x-1))$

$\overset{dist}{=} (x-1)(x+2)(x+2-x+1) = \boxed{3(x-1)(x+2)}$

(e) $x^5 - x$

$\overset{GCF}{=} x(\underbrace{x^4-1})$

$A^2 - B^2$ $\quad A = x^2,\ B = 1$

$x^4 - 1$

$\overset{A^2-B^2}{=} x\,(\underbrace{x^2-1})(\underbrace{x^2+1}_{8(b)})$ $\quad A^2 - B^2$

$= \boxed{x(x-1)(x+1)(x-i)(x+i)}$

9. Find a complete factorization of

$$f(x) = x^4 - x^3 - x - 1$$

given that $x = i$ and $x = -i$ are zeros. You must use the division algorithm.

By factor theorem

$$(x-i)(x-(-i)) = (x-i)(x+i) \overset{A^2-B^2}{=} (x^2 - i^2)$$

$$= (x^2 - (-1))$$

$$= x^2+1 \quad \text{is a factor.}$$

Use division algorithm.

$$\begin{array}{r|l} & x^2 - x - 1 \\ \hline x^2+1 & x^4 - x^3 + 0x^2 - x - 1 \\ & -(x^4 + x^2) \\ \hline & 0 - x^3 - x^2 - x \\ & -(-x^3 - x) \\ \hline & 0 - x^2 + 0 - 1 \\ & -(-x^2 - 1) \\ \hline & 0 \end{array}$$

So $f(x) = (x-i)(x+i)(x^2 - x - 1)$

$$= \boxed{(x-i)(x+i)\left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right)}$$

Quadratic formula

$a = 1, \quad b = -1, \quad c = -1$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\cdot 1 \cdot (-1)}}{2\cdot 1}$$

$$= \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

10. **Multi-Level Marketing (Exponential Growth)** Suppose your friend is trying to convince you to join a MLM network to sell products. Here are the conditions to join:

- **Membership**: You must pay USD \$120 yearly to stay as a member. This fee is meant "to cover" the network giving you products to sell.
- **Compensation plan**: You will not get paid for selling products **until** you recruit 3 or more members. After that, you will only get 15% of your recruits' profits.

For the rest of this problem, assume every member in this network recruits **only three people**.

(a) This network began with one person recruiting three people, then those three people recruiting three more for 9, and so on. We can represent this relationship with a function $f(x)$ where x is the "generation" you were recruited at and $f(x)$ represents the total number of people in generation x. For example,

i. $f(1) = 1$ because first generation only had one person

ii. $f(2) = 3$ since three people were recruited

iii. $f(3) = 9$ since those last three people recruited 3 people only.

Find a explicit formula for $f(x)$ representing an arbitrary generation x. Justify why your formula is correct.

generation # of people

$1 \longrightarrow 1 = 3^0$

$2 \longrightarrow 3 = 3^1$

$3 \longrightarrow 9 = 3^2$

$4 \longrightarrow 27 = 3^3$

so $\boxed{f(x) = 3^{x-1}}$

(b) Suppose this network is only taking place in San Luis Obispo, where for the sake of assumption there is 364 **total** people. How many generations will it take for everyone in SLO to be recruited if you're only allowed to recruit from SLO?

$f(1) + f(2) + f(3) + f(4) + f(5) + f(6)$

$= 3^0 + 3^1 + 3^2 + 3^3 + 3^4 + 3^5$

$= 1 + 3 + 9 + 27 + 81 + 243 = 364$

Six total generations

(c) Based on part (b), how many people are in the final generation?

$$f(5) = 243$$

(d) Based on your answer in part (b) and (c), what percent of people in SLO are in the final generation?

$$\frac{243}{364} \approx 66.75\%$$

$$\begin{array}{r} 364 \\ \times\ 6 \\ \hline 2184 \end{array} \qquad \begin{array}{r} 364 \\ \times\ 7 \\ \hline 2548 \end{array}$$

$$\begin{array}{r} 0.6675 \\ 364\,\overline{)\,243.0000} \\ -2184 \\ \hline 2460 \\ -2184 \\ \hline 2760 \\ -2548 \\ \hline 2100 \end{array}$$

(e) Explain why the percentage in part (d) shows that the majority of multi-level marketing members lose out on any profits.

The majority of members are in the final generation. These people recruited nobody. Yellow highlighted section on previous page implies these 66.75% will not make any money.

(f) Armed with this information, if you were to join a MLM, what do you need to watch out for?

Don't join a MLM. Recruiters target college students who don't know the vast majority of people make less than three digits USD per year.

source: https://www.amway.com/en_US/income-disclosure

11. Answer the following:

(a) One recurring theme throughout this quarter is given a problem like $x^2 = c$, the two answers that solve this equation are $x = \pm\sqrt{c}$. In English, explain why these two solutions are valid for the equation.

Because $(+\sqrt{c})^2 = (c^{\frac{1}{2}})^2 = c$ ✓

$(-\sqrt{c})^2 = (-1 \cdot c^{\frac{1}{2}})^2 = (-1)^2 \cdot (c^{\frac{1}{2}})^2 = c$ ✓

(b) Given the function $g(x) = -\log(-2x+4)$, describe **in English or with rules** the transformations needed to start from $f(x) = \log(x)$ to get to $g(x)$.

$g(x) = -\log(-2(x-2))$

1. $f(x) = \log(x)$
2. $h(x) = -f(x) = -\log(x)$
3. $k(x) = h(-x) = -\log(-x)$
4. $j(x) = k(2x) = -\log(-2x)$
5. $g(x) = j(x-2) = -\log(-2(x-2))$

(c) Determine if each expression is a factor of $\underbrace{x^4 - 2x^3 - 2x^2 + 5x - 2}_{P(x)}$:

i. $x-2$ $\quad P(2) = 16 - 16 - 8 + 10 - 2 = 0$ ✓

ii. $x+2$ $\quad P(-2) = 16 + 16 - 8 - 10 - 2 \neq 0$ ✗

iii. $x-1$ $\quad P(1) = 1 - 2 - 2 + 5 - 2 = 0$ ✓

iv. $x+1$ $\quad P(-1) = 1 + 2 - 2 - 5 - 2 \neq 0$

$x-2$ and $x-1$ are factors.

(d) Find the leading term and end behavior of $f(x) = 2(x-3)^2(x+4)^3$.

$f(x) = 2(x-3)(x-3)(x+4)(x+4)(x+4)$

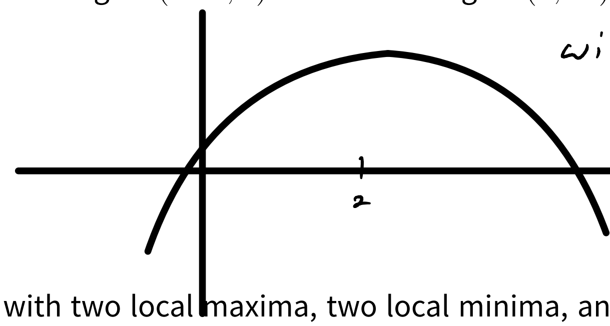
$= 2x^5 + \cdots$

leading term : $2x^5$

end behavior : $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$

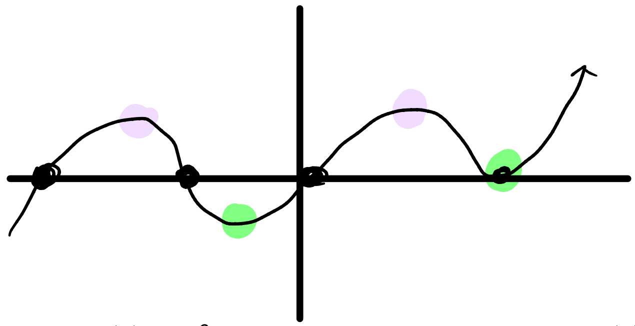
(e) If $f(x)$ is one-to-one, can it be increasing on $(-\infty, 2)$ and decreasing on $(2, \infty)$?

No.



will fail horizontal line test

(f) Sketch the graph of a function with two local maxima, two local minima, and 4 x-intercepts.



(g) If $f(x) = x + 1$ and $g(x) = x^2 - 1$, find and **fully expand** $-f(x)g(x)$.

$$-f(x)g(x) = -(x+1)(x^2-1)$$

$$\overset{dist}{=} -\left[(x+1)x^2 - (x+1)\cdot 1\right] \overset{dist}{=} -(x^3 + x^2 - x - 1)$$

$$= \boxed{-x^3 - x^2 + x + 1}$$

(h) Write down a formula of a quadratic that has vertex $(-2, 3)$.

$$f(x) = (x+2)^2 + 3$$

(i) Write down a degree 5 polynomial with four distinct zeroes.

$$f(x) = x(x-1)(x-2)(x-3)^2$$

12. Answer the following. Do not leave negative exponents.

(a) Simplify

$$\frac{1}{x-1}+\frac{1}{x+1}-\frac{2}{x}$$

LCD

$$= \frac{x(x+1)}{x(x+1)}\frac{1}{x-1}+\frac{x(x-1)}{x(x-1)}\frac{1}{x+1}-\frac{(x-1)(x+1)}{(x-1)(x+1)}\frac{2}{x}$$

frac law #1, dist law, A^2-B^2

$$= \frac{x^2+x}{x(x+1)(x-1)}+\frac{x^2-x}{x(x+1)(x-1)}-\frac{2(x^2-1)}{x(x+1)(x-1)}$$

frac law #3 dist

$$= \frac{x^2+x+x^2-x-2x^2+2}{x(x+1)(x-1)}$$

$$= \boxed{\frac{2}{x(x+1)(x-1)}}$$

(b) Simplify

$$x^2y\left(\frac{x+1}{y}\right)^{-2}\left(\sqrt{\frac{x}{y}}\right)^4$$

$$= x^2y\left(\frac{y}{x+1}\right)^2\left(\left(\frac{x}{y}\right)^{\frac{1}{2}}\right)^4$$

$\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}$ $(a^n)^m=a^{nm}$

$$= x^2y\frac{y^2}{(x+1)^2}\left(\frac{x}{y}\right)^2$$

frac law 1 $\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}$

$$= \frac{x^2y\cdot \cancel{y^2}}{(x+1)^2}\cdot\frac{x^2}{\cancel{y^2}}$$

frac law 1, 5

$$= \frac{x^2x^2y}{(x+1)^2}$$

$a^na^m=a^{n+m}$

$$= \boxed{\frac{x^4y}{(x+1)^2}}$$

(c) If

$$f(x) = x^2 + 1 \qquad g(x) = 2x^3 \qquad h(x) = 2x - 1 \qquad k(x) = 6x^2$$

fully expand and simplify the following expressions:

i. $f(x)g(x) + h(x)k(x)$

$$= (x^2+1)\,2x^3 + (2x-1)\,6x^2$$

$$\overset{\text{dist}}{=} 2x^5 + 2x^3 + 12x^3 - 6x^2$$

$$= \boxed{2x^5 + 14x^3 - 6x^2}$$

ii. $\dfrac{g(x)f(x) - k(x)h(x)}{[k(x)]^2} = \dfrac{2x^3(x^2+1) - 6x^2(2x-1)}{(6x^2)^2}$

$$\overset{\text{dist law}}{=} \frac{2x^5 + 2x^3 - 12x^3 + 6x^2}{6^2 \cdot (x^2)^2}$$

$(a \cdot b)^n = a^n b^n$

$(a^n)^m = a^{nm}$

$$= \boxed{\frac{2x^5 - 10x^3 + 6x^2}{36x^4}}$$